ON THE REPETITION OF ANGLES.

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(Read 26th April, 1892.)

The method of repetition admits of an almost unlimited accuracy whenever the quantity to be measured can be added to itself, without any error in the juxtaposition, until some multiple is found to coincide very nearly with a multiple of the unit employed. Some of the results obtained by the use of the method, such as the ratio of the mean solar day to the mean sidereal, or that of the mean motions of the sun and moon are known, in consequence, with a precision probably never equalled in direct measurements, few of which are correct to more than six significant figures. The pendulum, likewise, is an instance of a measuring instrument which owes its high perfection to the fact that it admits of indefinite repetition without introducing error in the adding up of the separate beats.

The application of this principle to the measurement of angles first proposed by Tobias Mayer in 1752, was embodied by Borda in the repeating circle, an instrument which soon became extensively adopted in Continental Europe whenever precise measurements were aimed at.

Theoretically, the advantages of the method are unquestionable when use is made of badly divided instruments that cannot be read with the requisite accuracy. For it may be shown that the probable error ε of an angle obtained by n repetitions and the two end readings is, calling α the probable error of a bisection and β that of a reading,

$$\varepsilon = \pm \sqrt{\frac{2\alpha^2}{n} + \frac{2\beta^2}{n^2}}$$

while the probable error of an angle obtained by n independent observations is

$$\varepsilon' = \pm \sqrt{\frac{2\alpha^2 + 2\beta^2}{n}}.$$

The greater accuracy of the first method over the second is thus measured by

$$\varepsilon' - \varepsilon = \frac{\dfrac{n-1}{n}\,\beta^2\sqrt{\dfrac{2}{n}}}{\sqrt{\alpha^2+\beta^2}+\sqrt{\alpha^2+\dfrac{\beta^2}{n}}},$$

a quantity which is always positive and increases with β, so that the advantage of the method increases with the imperfection of the instrument.

In practice, however, it was found that repeating instruments are affected with constant and periodic errors the source of which has not hitherto been satisfactorily explained. Col. A. R. Clarke (*Geodesy*, p. 18) summing up what is known on the subject says: "There are, however, other sources of error at work; the whole apparatus is not rigid as it is in theory supposed to be, and the play of the several axes doubtless affects the work with some constant error. Moreover it is a principle in observing generally, that to repeat the same observations over and over, under precisely the same circumstances, is a mere waste of time, the eye itself seems to take up under such circumstances a fixed habit of regarding the object observed, and that with an error which is for the time uniform. In some repeating circles a tendency has been found in the observed angle to continually increase or decrease as the number of repetitions was increased."

"W. Struve, in his account of his great arc in Russia, observes that if in measuring an angle the repetition be made first in the ordinary direction, and then again by reversing the direction of rotation of the circle, the two results differ systematically. Accordingly it became the practice to combine in measuring an angle rotations in both directions. Nevertheless there was no certainty that even then the error was eliminated, and the method of repetition was soon abandoned."

It remains difficult to understand why personal errors should become larger or more variable when bisecting a signal for the purpose of repetition than they are when the same operation is performed for the purpose of reiteration, and it is equally obscure why the play of the axes should be greater. No doubt, the play of the axes would be more marked in an instrument like the repeating circle with its multiplicity of axes and complication of parts, but it is extremely small in a good theodolite, and there is no apparent reason why it should vary with the nature of the observations.

The repeating circle being a deservedly obsolete instrument the errors by which it is affected need not be considered here. Besides, the classical researches on repetition, namely those of Struve (*Astron. Nachr.*, No. 47, 1824), and of Bessel (*Astron. Nachr.*, No. 256, 1834, and *Gradmessung in Ostpreussen*, p. 73) had reference to observations taken with repeating theodolites, and it was the unsatisfactory character of the results obtained with theodolites that led ultimately to the abandonment of the method.

Struve gives the following example of the differences obtained by using contrary rotations in the measurement of angles :

With the graduation.	Against the graduation.	Diff.	Mean angle.
° ′ ″	° ′ ″	″	
30·48·35·5	30·48·34·7	—0·8	30·48·35·1
64·27·25·2	64·27·23·0	—2·2	64·27·24·1
31·51·26·7	31·51·24·5	—2·2	31·51·25·6
73·22·38·1	73·22·34·2	—3·9	73·22·36·15
159·29·58·8	159·29·57·7	—1·1	159·29·58·25
360· 0· 4·3	359·59·54·1	—2·04 (mean)	359·59·59·2

The differences between angles measured with and against the graduation may be positive or negative. The table below gives, according to Jordan (*Handbuch der Vermessungskunde*, p. 270.) the differences found in the angles of the triangulation of Baden, carried between the years 1823–1852. The number of repetitions was usually six ; the number of measures varied from one to four and was not always the same for both series.

KLOSE'S OBSERVATIONS.

+4″·4	−0″·1	+3″·5	+2″·4	+1″·2	−0″·9
0·0	+0·5	+0·5	+1·5	−2·0	−1·6
0·0	−3·1	+0·4	+0·5	−0·1	−4·0
−4·8	+0·1	+1·6	+1·1	−2·0	−1·3
−3·4	−1·5	−4·0	+2·2	−4·0	−1·8
−1·2	0·0	+2·0	0·0	−1·6	
+1·1	+2·2	−1·4	+0·8	−2·5	
+1·9	+4·1	−2·3	+0·1	+0·2	

II. ROST'S OBSERVATIONS.

−2·6	+1·8	+3·8	−1·6	+0·3	+1·0
+0·7	+3·2	+1·5	−1·1	−0·3	+0·2
					+0·3

III. RHEIMER'S OBSERVATIONS.

−3·7	−2·0	+0·8	−2·3	−1·0	−2·0
0·0	−3·7	−0·4	−2·4	−1·3	−2·0
−0·2	−1·0	+0·4	−0·6	−2·0	−1·6
−0·6	−0·6	+0·1	−3·1	−5 0	−0·5
0·6	−3·1	+1·3	−2·2	+1·3	

Towards the end of 1890, I had occasion to carry a small secondary triangulation, over portions cf the districts of George and Knysna, to connect a few detached surveys and serve as a basis for subsequent work. The only instrument available for the purpose (a six-inch Everest theodolite by Troughton & Simms, read by two verniers to 20 seconds), although excellent for ordinary work, gave results that were not altogether satisfactory in a triangulation with sides averaging five or six miles. The method of repetition was then tried experimentally, but it was soon found that the instrument showed, to a curious degree, the peculiarities which ordinarily vitiate results obtained by repetition : There was an error in the angles, constant for a definite origin, but varying with the origin from a positive to a negative value ; there was, in addition, a periodic error, which, although it was to a great extent masked by the large mean error of each reading, seemed to vary with the number of the repetitions ; the direction of rotation had also a marked influence upon the resulting value.

These discrepancies could not be attributed in any way to the "back lash" of the tangent screws which was entirely taken up by unusually strong flatsprings of German silver, and their increase with the difference in altitude of the stations observed appeared to point to the inclination between the upper and lower vertical axes (a conspicuous defect of some theodolites) as being the chief cause of their occurrence in the observations.

In an angle obtained by repetition, the error due to imperfect adjustment of the instrument will be made up of the sum of the errors of each elementary angle divided by the number of repetitions. In each angle, it will consist of (a) error of collimation, (b) error due to inclination of the horizontal axis. The first, which is expressed by $\pm\, k''$ (cosec z_2 — cosec z_1), remains constant for each angle, and its amount may therefore be simply added to the value obtained by repetition. The second may be determined by reading on a striding level the inclination of the horizontal axis in each different position, adding the corrections for each angle, and dividing by the number of repetitions. Or it may be calculated directly. It consists of the sum of the errors due to (1) the inclination of the horizontal axis relatively to the vertical axis, (2) the inclination of the vertical axis itself. Error (1) is given by

$$\pm\, h'' (\cot z_2 - \cot z_1)$$

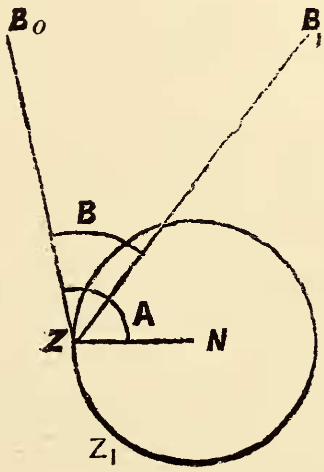


Fig. 1.

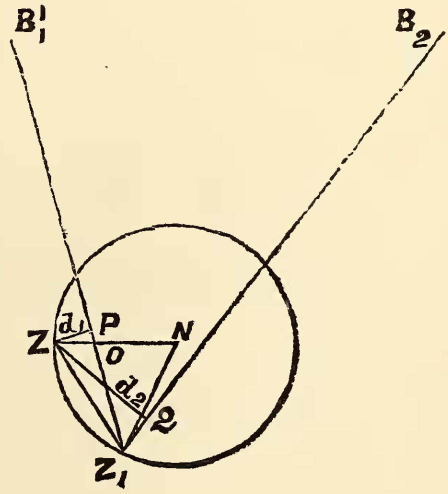


Fig. 2.

To find error (2), let, in fig 1, Z be the point of a sphere where it is intersected by the upper vertical axis when levelled truly vertical, and N, the intersection of the lower vertical axis. By rotation about the lower axis, Z will describe the small circle ZZ_1Z. Call B the angle B_0ZB_1 under repetition, A the angle B_0ZN, and $i'' = ZN$, the inclination between the vertical axes.

In fig. 2, let $B'_1Z_1B_2$ be the angle B at the second repetition ; it is then equal to ZNZ_1. Z remaining the vertical throughout, draw the perpendiculars $ZP = d_1$ and $ZQ = d_2$ equal, respectively, to the inclination of the horizontal axis due to that of the vertical axis, for each of the directions B'_1, B_2 observed. The correction to the second angle will therefore be

$$x_2 = d_2 \cot z_2 - d_1 \cot z_1$$

In the small spherical triangle ZZ_1P, right angled at P,

$$d_1 = ZZ_1 \sin ZZ_1P$$

And in triangle ZNZ_1

$$zz_1 = 2\, i'' \sin \frac{B}{2};$$

therefore

$$d_1'' \cot z_1 = 2\, i'' \sin \frac{B}{2} \sin ZZ_1P \cot z_1.$$

But from triangle ZOZ_1, in which $O = A$, and $Z = \frac{\pi}{2} - \frac{B}{2}$,

$$ZZ_1P = \frac{\pi}{2} - A + \frac{B}{2},$$

so that

$$d_1'' \cot z_1 = 2i'' \sin \frac{B}{2} \cos \left(A - \frac{B}{2}\right) \cot z_1;$$

similarly

$$d_2'' \cot z_2 = 2i'' \sin \frac{B}{2} \cos \left(A - \frac{3B}{2}\right) \cot z_2,$$

and

$$x_2 = 2i'' \sin \frac{B}{2} \left[\cos \left(A - \frac{3B}{2}\right) \cot z_2 - \cos \left(A - \frac{B}{2}\right) \cot z_1\right].$$

Similarly, the error in the third angle will be

$$x_3 = 2i'' \sin \frac{2B}{2} \left[\cos \left(A - \frac{4B}{2}\right) \cot z_2 - \cos \left(A - \frac{2B}{2}\right) \cot z_1\right],$$

and in the nth angle

$$x_n = 2i'' \sin \frac{n-1}{2} B \left[\cos\left(A - \frac{n+1}{2} B\right) \cot z_2 - \cos\left(A - \frac{n-1}{2} B\right) \cot z_1\right].$$

Let

$$S_2 = \sin \frac{B}{2} \cos \left(A - \frac{3B}{2}\right) + \sin \frac{2B}{2} \cos \left(A - \frac{4B}{2}\right) + \ldots$$
$$+ \sin \frac{n-1}{2} B \cos \left(A - \frac{n+1}{2} B\right)$$

$$S_1 = \sin \frac{B}{2} \cos \left(A - \frac{B}{2}\right) + \sin \frac{2B}{2} \cos \left(A - \frac{2B}{2}\right) + \ldots$$
$$+ \sin \frac{n-1}{2} B \cos \left(A - \frac{n-1}{2} B\right).$$

Summing these series, it is found that

$$2S_2 = (n - 1) \sin \alpha - \frac{\sin (\alpha - n\beta) \sin (n - 1) \beta}{\sin \beta}$$

$$2S_1 = (n - 1) \sin A - \frac{\sin (A - n\beta) \sin (n - 1) \beta}{\sin \beta}$$

$$\text{in which } \beta = \frac{B}{2}, \text{ and } \alpha = A - B;$$

and the correction x to the angle obtained by repetition is

$$x = \frac{x_1 + x_2 + \dots + x_n}{n}$$

$$= \frac{i''}{n} (2S_2 \cot z_2 - 2S_1 \cot z_1).$$

It is seen from this formula that the correction will vary in magnitude and change in sign with different values of α, β, n, and z_1, z_2. By reversing the direction of rotation, it is only in particular cases that the mean will be freed of error. The method due to Struve thus fails in ordinary cases, although it sometimes may reduce the error very considerably (for instance when A is nearly equal to $\frac{B}{2}$, or $180° + \frac{B}{2}$, with little difference between z_1 and z_2). But by making A into $180° + A$, the sign of the correction is changed and the mean freed of the error due to the inclination between the axes without preventing the simultaneous elimination of the remaining instrumental errors by the method of reversal.

In order to compare calculated corrections with actual discrepancies, an angle was observed on the 20th Nov. 1890, at the station "Roode" between "Spijoenkop" (— 0° 51′ 20″ app. alt.) and "Hoogeberg" (+ 5° 59′ 30″ app. alt.) and two series, with the origins 0° and 180°, obtained. This angle was selected because it gave the largest difference in zenith distance in the survey, and would, on that account, make the character of the errors most apparent. The observations were :

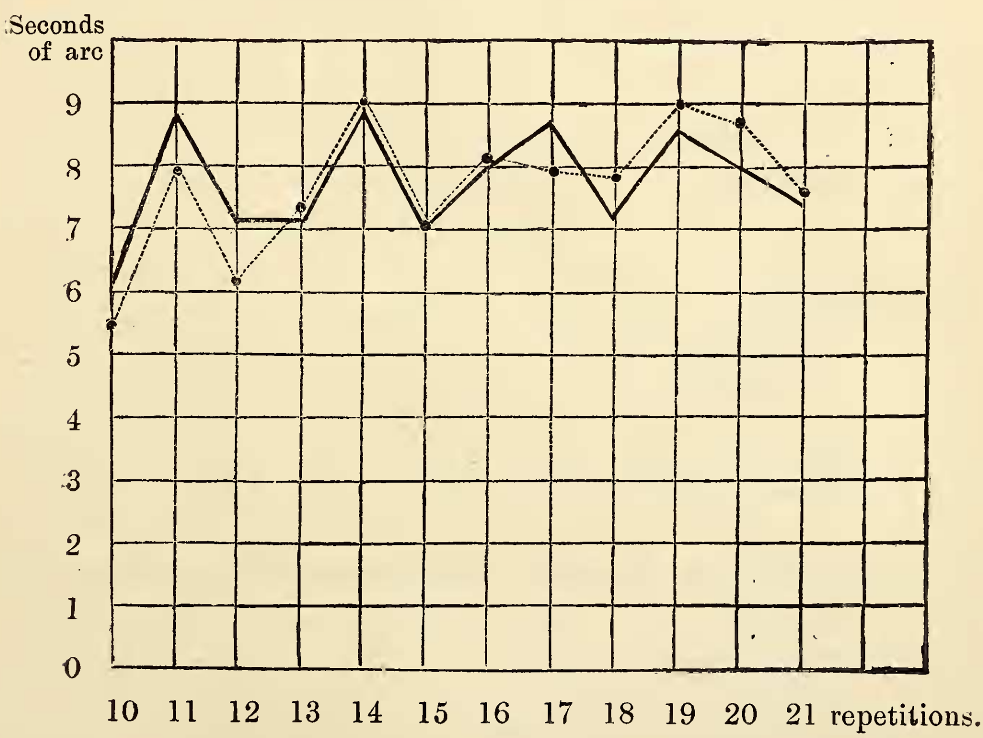
Number of repetitions.	Mean of Vernier Readings.		Resulting Angle.	
	Origin 0°	Origin 180°	Origin 0°	Origin 180°
	° ′ ″	′ ″	° ′ ″	′ ″
0	359 50 0	50 0	...	...
1	131 53 10	52 55	...	...
10	240 18 0	19 50	132 2 48·0	2 59·0
11	12 20 15	23 10	2 45·0	3 0·9
12	144 23 20	25 50	2 46·7	2 59·2
13	276 25 45	28 55	2 45·0	2 59·5
14	48 28 15	32 25	2 43·9	3 1·8
15	180 31 35	35 5	2 46·3	3 0·3
16	312 34 10	38 30	2 45·6	3 1·9
17	84 36 45	41 15	2 45 0	3 0·9
18	216 39 50	44 30	2 46·1	3 1·7
19	348 41 55	47 35	2 43·9	3 1·9
20	120 44 55	50 40	2 44·7	3 2·0
21	252 48 0	53 20	2 45·7	3 1·0

The instrumental errors of the 6-inch Everest theodolite used in the measurement were determined by obvious methods ; the inclination of the horizontal axis relatively to the upper vertical axis was found to be 50·″0 (right hand pivot highest) and that between the upper and lower vertical axes 4′ 26″ in the direction of division 93 (so that $A = 273°$.) The collimation was adjusted accurately enough to render its effect upon the observed angles practically insensible.

The calculation of the corrections from these data, by the formulae above, gave the results shown in this table :

Number of repetitions.	Correction for inclination of		Total.	Half of observed difference.
	vertical axis	horiz. axis		
	"	"	"	"
10	12·07	—5·99	6·07	5·5
11	14·78	—5·99	8·79	7·95
12	13·16	—5·99	7·17	6·25
13	13·17	—5·99	7·18	7·25
14	14·80	—5·99	8·81	8·95
15	12·98	—5·99	6·99	7· 0
16	14·01	—5·99	8·02	8·15
17	14·46	—5·99	8·47	7·95
18	13·12	—5·99	7·13	7·8
19	14·55	—5·99	8·56	9·0
20	14·05	—5·99	8·06	8·65
21	13·50	—5·99	7·51	7·65

The concordance of the calculated corrections with the observed differences between the series differing 180° in origin may be seen better in a diagram.



The dotted line represents half the observed differences, and the black line, the calculated corrections; their agreement, although satisfactory, would perhaps have been still closer if the inclinations between the several axes could have been more accurately determined and the pivot errors investigated.

As an example of the precision attainable with ordinary instruments of small size, by following the methods of elimination of error which have been pointed out, the data of a simple quadrilateral, the last in the small chain of triangles already referred to, are appended :

Observed Angles: Nov. 1890.

		$\frac{1}{w}$	No. of repetitions.
At Spijoenkop			
Portland Height ...	$0°\ 0'\ 0''\cdot 0$		
Rondevallei	$216\cdot 57\cdot\ 7\cdot 1+(1)$	9·43	10
Cornelis Kop ...	$278\cdot 29\cdot\ 0\cdot 7+(2)$	9·43	10
Roode	$311\cdot\ 9\cdot 21\cdot 7+(3)$	9·43	10
At Roode			
Spijoenkop	$0\cdot\ 0\cdot\ 0\cdot 0$		
Rondevallei	$44\cdot\ 7\cdot 16\cdot 3+(4)$	2·40	20
Cornelis Kop .	$93\cdot\ 9\cdot 12\cdot 1+(5)$	2·40	20
At Cornelis Kop			
Spijoenkop	$0\cdot\ 0\cdot\ 0\cdot 0$		
Rondevallei	$51\cdot 29\cdot 28\cdot 8+(6)$	9·43	10
Roode	$305\cdot 49\cdot 32\cdot 0+(7)$	2·40	20
At Rondevallei			
Touw's Berg ...	$0\cdot\ 0\cdot\ 0\cdot 0$		
Cornelis Kop ...	$58\cdot 24\cdot\ 1\cdot 8+(8)$	9·43	10
Roode	$83\cdot 42\cdot 12\cdot 4+(9)$	25·15	10 (*a*)
Spijoenkop	$125\cdot 22\cdot 39\cdot 9+(10)$	9·43	10

The length of the side Cornelis—Spijoenkop is 4066·540 Cape Roods.

(*a*) The observations at Rondevallei between Touw's Berg and Roode were made in a high wind, interrupted by rain, and the end reading taken in the dark after probable slight shifting of the vernier plate. The station could not conveniently be revisited, but a smaller weight was assigned to the angle in the calculations.

The reduction of these observations by least squares gave the following corrections, which enter in three angle, and one side equations :

$$
\begin{aligned}
&& {}^{\prime\prime} \\
(1) &= -0{\cdot}50 \\
(2) &= +0{\cdot}47 \\
(3) &= +0{\cdot}04 \\
(4) &= +0{\cdot}38 \\
(5) &= -0{\cdot}23 \\
(6) &= -0{\cdot}87 \\
(7) &= +0{\cdot}14 \\
(8) &= +0{\cdot}24 \\
(9) &= -1{\cdot}04 \\
(10) &= +0{\cdot}14
\end{aligned}
$$

Only a few of the other angles of the triangulation were obtained by repetition in the manner finally adopted in the last portion of the work, and the other corrections are thus not generally comparable ; correction (3) derived from an adjoining block is also $+0''{\cdot}04$, an agreement which must however be largely due to fortuitous causes.

The manner of observing was this :—The zero of the circle having been made to correspond nearly to one of the directions observed, the upper vertical axis was levelled vertical, by means of the level of the vernier plate, in two directions at right angles, regardless of other positions. For the reversed series, the telescope was turned half round the horizontal axis without changing the pivots, so as to eliminate in the mean, both the collimation error and that due to the inclination of the horizontal axis relatively to the upper vertical axis. By turning the circle 180°, the first zero was again taken, and by relevelling the upper vertical axis with the level in the two directions opposite those adopted in the first series, the residual level error was neutralised, as well as the error introduced by the inclination between the vertical axes. It may be noticed incidentally that this mode of elimination of uncorrected level error is also applicable to ordinary work by reiteration, provided the vertical axis is sufficiently true. To obviate so much as possible the play of the footscrews, a source of error which proved one of the most troublesome to avoid, both the circle and the vernier plate were uniformly turned in one direction until the direct series was completed; they were then uniformly turned in the contrary direction during the reversed series, but the angles measured, in both cases, with the graduation, that is starting from the left hand station at near zero. Care was also taken not to disturb the focus as the line of collimation did not coincide with the axis of the draw tube.

Observations were made only in favourable weather (except at Rondevallei), during the period of steadiness of the images ; generally between sunrise and nine in the morning and from 3·30 to sunset in the afternoon. Two end readings were taken and the mean of the results from 0 and n, and from 1 and $n+1$, repetitions, assumed to give the mean value. Thus with ten repetitions one-tenth was taken of the mean of the angles between 0 and 10 and between 1 and 11 repetitions.

These precautions, gradually introduced during the course of the earlier portion of the triangulation, were followed by the noteworthy increase in accuracy which is indicated below. It should however be also mentioned that the symmetry of the large stone beacons used in the first few triangles left much to be desired, an important source of error with sides of a few miles, whereas in the later portion of the work, good wooden signals were put up, chiefly to test the accuracy of the method.

First 2	triangles	mean close	$6''\!\cdot\!4$
Next 5	„	„	3·2
Last 6	„	„	1·3

The first two triangles, and two of the next five, were beyond the chain connecting the stations Hoogeberg and Belvidere of Capt. Bailey's survey which were used as a base, so that their lesser accuracy did not affect that of the connection.

In a repeated angle, the *à priori* probable error cannot be deduced from the agreement of the observations with their means, as in the case of an angle obtained by reiteration ; but if the observations at Roode between Spijoenkop and Hoogeberg, which have already been given, be taken singly, and also those of another set taken at the same station between Spijoenkop and Rondevallei, we get the results contained in the annexed table :

Spijoenkop Hoogeberg.	Error.	Square.	Spijoenkop Rondevallei.	Error.	Square.
° ′ ″	″		° ′ ″	″	
132 3 2·5	+ 8·5	72·25	43 7 30	+12·7	161·29
2 47·5	− 6·5	41·25	7 20	+ 2·7	7·29
2 52·5	− 1·5	2·25	7 20	+ 2·7	7·29
2 45	− 9·0	81	7 12·5	− 4·8	23·04
3 0	+ 6·0	36	7 15	− 2·3	5·29
3 0	+ 6·0	36	7 25	+ 7·7	59·29
3 0	+ 6·0	36	7 5	−12·3	151·29
2 40	−14·0	196	7 17·5	+ 0·2	0·04
3 10	+16·0	256	7 20	+ 2·7	7·29
2 35	−19·0	361	7 15	− 2·3	5·29
3 2·5	+ 8·5	72·25	7 22·5	+ 5·2	27·04
2 52·5	− 1·5	2·25	7 5·0	−12·3	151·29
2 54·0 (mean)		1192·25	7 17·3 (mean)		605·73

Taking the two sets together, in order to arrive at an estimate of the ordinary probable error of an observed angle, the sum of the squares is 1797·98, hence the probable error of a single determination (mean of direct and reversed) is

$$\pm \cdot 6745 \left(\frac{1797\cdot 98}{24-2}\right)^{\frac{1}{2}} = \pm 6''\cdot 10$$

If α the probable error of a bisection be taken at $\pm 0''\cdot 80$ for the instrument used, β the probable error of a reading becomes $\pm 6''\cdot 04$ (α being small in comparison to β, a small error in the estimation of the first scarcely affects the result.)

The comparison of the corrections applied to the angles of the quadrilateral with the probable errors calculated, from the above values of α and β, for angles obtained by direct and reversed series and double end readings is shown below :

	Calculated prob. error.	Mean of corrections.
	″	
10 repetitions	± 0·49	0·49 (mean of 7)
20 repetitions	± 0·28	0·25 (mean of 3)

The corrections found requisite are thus well within the limits indicated by theory, and it is not unlikely that the unsatisfactory hiatus which has hitherto prevailed between the theory and the practice of repetition may be made to disappear by the adoption of suitable instruments and proper methods of observing.

It is, nevertheless, very improbable that repetition will again supersede reiteration for work of the highest precision in which first-rate instruments are used. When β the probable error of a reading becomes small compared to α the error of a bisection, there is little difference in theoretical accuracy between the two methods, especially if account be taken of the rapid elimination, by the ordinary one, of the systematic errors of graduation. There remain important advantages on the side of the present method: it gives the means of better eliminating, by multiplying the observations on different days and under different conditions, the usually much more serious errors introduced by the atmosphere, and since with the perfection of modern instruments the terms involving β become nearly as small or smaller than the terms involving α, it usefully allows of several bisections with an eye-piece micrometer for each reading of the circle.

But with smaller triangles, when the accuracy aimed at is well below the limit practically imposed by the presence of the atmosphere, and a number of observations are still required with the instrument used, repetition often affords the quickest method of measurement of the horizontal angles. When the instrument is small and the error of a reading large in consequence, the advantage may become very great. With the theodolite for which $\alpha = \pm 0''\cdot 80$ and $\beta = \pm 6''\cdot 04$ the probable error of an observed angle was $\pm 0''\cdot 49$ with ten repetitions and $\pm 0''\cdot 28$ with twenty, and these values were borne out by the smallness of the corrections required to satisfy the geometrical conditions of the triangulation. To have obtained the same accuracy by reiteration 150 duplications in the first case, and 490 in the second, would have been needed. As accurate work can be done only during a few hours a day, it is seen that a precision which was attained in a single visit to each station would have been practically out of reach with reiteration.

The method of repetition may thus be often employed with advantage in the secondary triangulation of a country. It ought, in particular, to become valuable to ordinary surveyors who are called upon to provide, by triangulation, a framework for extensive property or topographical surveys, because it allows of the use of small instruments for the most accurate work likely to be required in practice, two material axes without play, and strong springs to the tangent screws, being all that is essential to render a theodolite suitable for the application of the principle.